Exam in Circuit QED, M2 ICFP

The flux qubit

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Documents are authorized, but not computers, tablets, phones or any device that can connect to the internet.

In this exercise we will analyze an emblematic superconducting qubit: the flux qubit. This qubit was proposed at Delft in 1999 [Mooij *et al.*, Science, vol 285 p 1035], and demonstrated in 2003 [Chiorescu *et al.*, Science, vol 299, p. 1869]. Notably, the Canadian quantum computing company D-Wave manufactures chips of thousands of flux qubits for quantum simulations [King *et al.*, Science, vol 388, p. 199]. The circuit is displayed in Fig. 1. The goal here is to analyze the circuit and compute its low-energy spectrum.

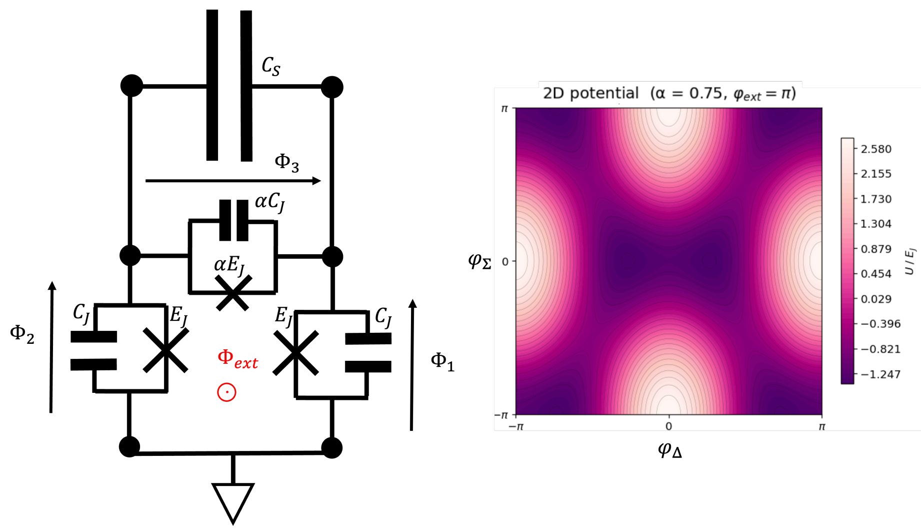

Figure 1: (Left) circuit diagram of a heavy 3-junction flux qubit. Two nominally identical Josephson junctions of energy E_J and capacitance C_J, are connected through a third junction of energy $\alpha \times E_J$ and capacitance $\alpha \times C_J$, where $0 < \alpha < 1$. The three junctions form a loop threaded by a time-independent flux Φ_{ext}. The third junction is also capacitively loaded by a large capacitor of capacitance $C_S \gg C_J$. (Right) Potential energy for $\alpha = 0.75$ and $\varphi_{\mathrm{ext}} = \pi$.

1. Consider the circuit displayed in Fig. 1. Write down the constraint that links the degrees of freedom Φ_1, Φ_2, Φ_3 and the static external flux Φ_{ext}.
2. Compute the potential energy $U(\Phi_1, \Phi_2)$.
3. Compute the kinetic energy $T(\dot{\Phi}_1, \dot{\Phi}_2)$.
4. Express U and T in terms of the common and differential fluxes:

$$\begin{aligned} \Phi_\Sigma &= (\Phi_1 + \Phi_2)/2 \\ \Phi_\Delta &= (\Phi_1 - \Phi_2)/2\,. \end{aligned}$$

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5. We recall that the phase φ corresponding to flux Φ is defined as $\varphi = 2\pi\Phi/\Phi_0$, with Φ_0 being the superconducting flux quantum.

   Show that the circuit Hamiltonian takes the form

$$\begin{aligned}\hat{H} &= 4E_{C_\Sigma}N_\Sigma^2 + 4E_{C_\Delta}N_\Delta^2 + U(\varphi_\Sigma, \varphi_\Delta) \\ U(\varphi_\Sigma, \varphi_\Delta) &= -E_J\left(2\cos(\varphi_\Delta)\cos(\varphi_\Sigma) + \alpha\cos(2\varphi_\Delta - \varphi_{\text{ext}})\right)\ ,\end{aligned}$$

   and derive the expressions for $E_{C_\Sigma}, E_{C_\Delta}$. Simplify these expressions in the limit $C_S \gg C_J$.

6. For the remainder of this exercise, we will focus on the bias point $\varphi_{\text{ext}} = \pi$. What are the symmetries of the potential $U(\varphi_\Sigma, \varphi_\Delta)$ (i.e. inversion symmetries, translational symmetries, mirror symmetries, etc) ?

7. In order to gain insight on the lowest eigenstates of H, we will locate the minima of the potential U.

   Justify that we may limit our search for potential minima along the segment $\varphi_\Sigma = 0$. For the remainder of the exercise, we will freeze $\varphi_\Sigma = 0$ and treat this as 1D problem of the variable φ_Δ.

8. What values of φ_Δ are extrema of U along the segment $\varphi_\Sigma = 0$? Verify if these extrema are minima or maxima. Separate the cases $\alpha > 1/2$ and $\alpha < 1/2$. What does crossing the value $\alpha = 1/2$ remind you of ?

9. Plot the 1D potential $\varphi_\Delta \to U(\varphi_\Sigma = 0, \varphi_\Delta)$ in the two cases $\alpha < 1/2$ and $\alpha > 1/2$. For each case, express the persistent current circulating in the loop for each potential minimum.

10. Let's first assume $\alpha \ll 1/2$ and $E_J \gg E_{C_\Delta}$. By analogy to a circuit derived in class, describe the spectrum of this circuit.

11. In the case where $\alpha > 1/2$, represent qualitatively the two lowest eigenfunctions of this circuit for "small" and "large" C_S. These generally serve as the flux qubit computational states.