Exam in Circuit QED, M2 ICFP
Inductive qubit-resonator coupling

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During the lectures we studied the capacitive coupling between a Transmon qubit and a resonator. In this exercise we will instead analyze the inductive coupling between a Fluxonium qubit and a resonator. Its circuit is displayed in Fig. 1. The goal here is to compute the inductively induced coupling strength, and compare it to the capacitive one. This type of circuit is extensively employed in the field of superconducting qubits [Nature 508, 369–372 (2014), PRB 94, 144507 (2016), PRX 12, 021002 (2022)].

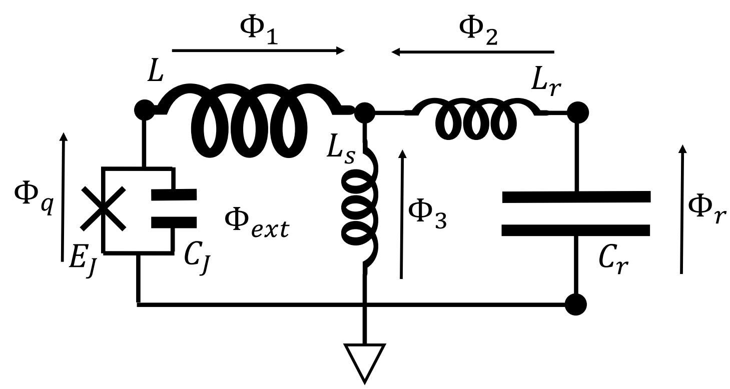

Figure 1: Circuit diagram of a Fluxonium qubit inductively coupled to a resonator.

1. Consider the circuit displayed in Fig. 1. Write down the three constraints that link the degrees of freedom $\Phi_1, \Phi_2, \Phi_3, \Phi_q, \Phi_r$ and the static external flux Φ_{ext}.

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This circuit contains 2 loop constraints and one node constraint. The two loop constraints yield

$$\begin{aligned} \Phi_r + \Phi_2 - \Phi_3 &= 0 \quad (1) \\ \Phi_3 - \Phi_1 - \Phi_q &= \Phi_{\text{ext}} \ , \quad (2) \end{aligned}$$

and current conservation in the central node yields

$$\Phi_1/L + \Phi_2/L_r + \Phi_3/L_s = 0 \ . \quad (3)$$

2. Justify that in the limit where $L_r \sim L_s \ll L$, the current conservation law at the central node my be simplified to

$$\Phi_3/L_s + \Phi_2/L_r = 0 \ .$$

When current flowing on the right branch (inductance L_r and capacitance C_r) arrives at the central node, it may split in two paths. In the limit where $L_s \ll L$, the path through L_s has a much lower impedance and so the current will preferably flow through the central branch. Current conservation implies

$$\Phi_3/L_s + \Phi_2/L_r = 0 \ . \quad (4)$$

Note that one may reach the same result by taking the limit $L \to +\infty$ in Eq. (3).

Remark: The rigorous approach would be to keep the full current conservation law of Eq. (3) and solve question 3 by inverting a 3x3 matrix. The intention here was to crudely simplify Eq. (3) to Eq. (4) in order to simplify the following question. One may worry that we have neglected first order terms in $L/L_{r,s}$ and that this will false the final result, but one can verify that the rigorous approach gives the same result as the one proposed here up to second order terms in $L/L_{r,s}$.

3. Express Φ_1, Φ_2, Φ_3 as a function of $\Phi_r, \Phi_q, \Phi_{\text{ext}}$.

We will now invert Eqs (1)(2)(4) to express Φ_1, Φ_2, Φ_3 as a function of Φ_r, Φ_q. From Eq. (4), we have $\Phi_3 = -(L_s/L_r)\Phi_2$. Injecting this in Eq. (1), we get

$$\Phi_2 = -\frac{L_r}{L_s + L_r}\Phi_r \ , \quad (5)$$

and hence

$$\Phi_3 = \frac{L_s}{L_s + L_r}\Phi_r \ . \quad (6)$$

Injecting this last expression in Eq. (2) yields

$$\Phi_1 = \frac{L_s}{L_s + L_r}\Phi_r - \Phi_q - \Phi_{\text{ext}} \ . \quad (7)$$

4. Compute the potential energy $U(\Phi_r, \Phi_q)$. Simplify the expression in the limit where $L_r \sim L_s \ll L$. Use the notation $\tilde{L}_r = L_r + L_s$, and the change of variables $\Phi_q \to \Phi_q - \Phi_{\text{ext}}$.

   The potential energy is

   $$U = \Phi_1^2/2L + \Phi_2^2/2L_r + \Phi_3^2/2L_s - E_J \cos(2\pi\Phi_q/\Phi_0) \, ,$$

   where Φ_0 is the superconducting flux quantum. After substituting $\Phi_{1,2,3}$ by their expressions as a function of $\Phi_{r,q}$ we get:

   $$\begin{aligned} U &= \frac{1}{2L}\left(\frac{L_s}{L_s+L_r}\Phi_r - \Phi_q - \Phi_{\text{ext}}\right)^2 + \frac{1}{2(L_r+L_s)}\Phi_r^2 \\ &- E_J\cos(2\pi\Phi_q/\Phi_0) \, . \end{aligned}$$

   Using the notation $\tilde{L}_r = L_r + L_s$, and the change of variables $\Phi_q \to \Phi_q - \Phi_{\text{ext}}$, we get

   $$\begin{aligned} U &= \frac{1}{2L}\left(\frac{L_s}{\tilde{L}_r}\Phi_r - \Phi_q\right)^2 + \frac{1}{2\tilde{L}_r}\Phi_r^2 \\ &\quad -E_J\cos(2\pi(\Phi_q - \Phi_{\text{ext}})/\Phi_0) \\ &= \frac{1}{2L}\Phi_q^2 + \frac{1}{2\tilde{L}_r}(1 + \frac{L_s^2}{L\tilde{L}_r})\Phi_r^2 - \frac{L_s}{L\tilde{L}_r}\Phi_r\Phi_q \\ &\quad -E_J\cos(2\pi(\Phi_q - \Phi_{\text{ext}})/\Phi_0) \\ &\approx \frac{1}{2L}\Phi_q^2 + \frac{1}{2\tilde{L}_r}\Phi_r^2 - \frac{L_s}{L\tilde{L}_r}\Phi_r\Phi_q \\ &\quad -E_J\cos(2\pi(\Phi_q - \Phi_{\text{ext}})/\Phi_0) \, . \end{aligned}$$

   In the last expression we have used the fact that $L \gg L_s \sim L_r$.

5. Compute the kinetic energy $T(\dot{\Phi}_r, \dot{\Phi}_q)$ of this circuit. Deduce the expression of the charge Q_r, Q_q.

   The kinetic energy reads

   $$T = \frac{C_J}{2}\dot{\Phi}_q^2 + \frac{C_r}{2}\dot{\Phi}_r^2 \, .$$

   The charges are defined from the Lagrangian $L = T - U$ as $Q_q = \frac{\partial L}{\partial \dot{\Phi}_q} = C_J\dot{\Phi}_q$ and $Q_r = \frac{\partial L}{\partial \dot{\Phi}_r} = C_r\dot{\Phi}_r$.

6. Derive the quantum Hamiltonian of this circuit. Comment on the different parts of this Hamiltonian.

   The Hamiltonian is defined as $H(Q_{q,r}, \Phi_{q,r}) = Q_q\dot{\Phi}_q + Q_r\dot{\Phi}_r - L(\dot{\Phi}_{q,r}, \Phi_{q,r})$, where $\dot{\Phi}_{q,r}$ must be expressed as a function of $Q_{q,r}$. Variables are then promoted to operators, we find

$$\begin{aligned}\hat{H} &= \frac{\hat{Q}_q^2}{2C_J} + \frac{\hat{\Phi}_q^2}{2L} - E_J\cos(2\pi(\hat{\Phi}_q - \Phi_{\rm ext})/\Phi_0) \\ &+\frac{\hat{Q}_r^2}{2C_r} + \frac{\hat{\Phi}_r^2}{2\tilde{L}_r} \\ &-\frac{L_s}{L\tilde{L}_r}\hat{\Phi}_r\hat{\Phi}_q \ .\end{aligned}$$

The first line corresponds to the Fluxonium Hamiltonian that is a widely used superconducting qubit. The second line corresponds to a quantum harmonic oscillator, often used as a readout resonator. The last line corresponds to a coupling term between the Fluxonium and the resonator.

7. Express the Hamiltonian as a function of annihilation and creation operators. You may use the notation $\hat{a}, \hat{a}^\dagger$ for the resonator mode, and $,\hat{b}^\dagger$ for the qubit mode.

   Let us introduce raising and lowering operators such that

$$\begin{aligned}\hat{\Phi}_r &= \Phi_{zpf,r}(\hat{a}+\hat{a}^\dagger) \\ \hat{Q}_r &= Q_{zpf,r}(\hat{a}-\hat{a}^\dagger)/i \ ,\end{aligned}$$

   where $\Phi_{zpf,r} = \sqrt{\hbar Z_r/2}$ and $Q_{zpf,r} = \sqrt{\hbar/2Z_r}$ where $Z_r = \sqrt{\tilde{L}_r/C_r}$. Similarly

$$\begin{aligned}\hat{\Phi}_q &= \Phi_{zpf,q}(\hat{b}+\hat{b}^\dagger) \\ \hat{Q}_q &= Q_{zpf,q}(\hat{b}-\hat{b}^\dagger)/i \ ,\end{aligned}$$

   where $\Phi_{zpf,q} = \sqrt{\hbar Z_q/2}$ and $Q_{zpf,q} = \sqrt{\hbar/2Z_q}$ where $Z_q = \sqrt{L/C_J}$. Injecting these expressions in the Hamiltonian, we find

$$\begin{aligned}\hat{H} &= \hbar\Omega_q\hat{b}^\dagger\hat{b} - E_J\cos(2\pi(\Phi_{zpf,q}(\hat{b}+\hat{b}^\dagger) - \Phi_{\rm ext})/\Phi_0) \\ &+\hbar\Omega_r\hat{a}^\dagger\hat{a} \\ &-\frac{L_s}{L\tilde{L}_r}\Phi_{zpf,q}\Phi_{zpf,r}(\hat{a}+\hat{a}^\dagger)(\hat{b}+\hat{b}^\dagger) \ ,\end{aligned}$$

   where the frequencies $\Omega_q = 1/\sqrt{LC_J}$ and $\Omega_r = 1/\sqrt{\tilde{L}_rC_r}$.

8. Express the coupling strength g between the qubit and the resonator as a function of L_s and the resonator and qubit resonance frequencies and impedances.

   The coupling strength g is identified as the prefactor to the term in $(\hat{a}+$

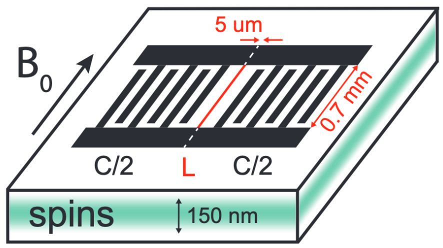

Figure 2: Low impedance resonator optimized for enhanced inductive coupling (here to spins). Image extracted from [Bienfait et. al. Nature Nanotechnology volume 11, pages 253–257 (2016)]

$\hat{a}^\dagger)(\hat{b}+\hat{b}^\dagger)$. We have

$$\begin{aligned} \hbar g &= \frac{L_s}{L\tilde{L}_r}\Phi_{zpf,q}\Phi_{zpf,r} \\ &= \frac{L_s}{L\tilde{L}_r}\sqrt{\hbar Z_q/2}\sqrt{\hbar Z_r/2}\,. \end{aligned}$$

Note that $L = Z_q/\Omega_q$ and $\tilde{L} = Z_r/\Omega_r$. hence

$$g = \frac{L_s}{2}\Omega_q\Omega_r\frac{1}{\sqrt{Z_qZ_r}}\,.$$

9. Comment on the dependence of g on the resonator impedance, and how it compares to the case of capacitive coupling studied in class. Provide a qualitative drawing of a resonator that favors inductive coupling.

   Note that $g \propto 1/\sqrt{Z_r}$. This is reversed with respect to the capacitive coupling between a transmon and a resonator where the coupling is proportional to $\sqrt{Z_r}$. Hence in order to optimize the inductive coupling, one needs a low impedance resonator, that is low $\tilde{L}_r$ and a large C_r. In practice, this is obtained by shunting a short wire by a large interdigitated capacitor. This is implemented in the CEA Saclay group to maximize the inductive coupling of a resonator to spins (see Fig.2).

10. (bonus) Provide a strategy to study the *non-linear* coupling between the qubit and resonator induced by the Josephson junction.

    One strategy, described in [PRB 94, 144507 (2016)], is to introduce modes that diagonalize the linear Hamiltonian

$$H_0/\hbar = \Omega_q\hat{b}^\dagger\hat{b} + \Omega_r\hat{a}^\dagger\hat{a} - g(\hat{a}+\hat{a}^\dagger)(\hat{b}+\hat{b}^\dagger)\,.$$

    The linear Hamiltonian will then take the form

$$H_0 \to \Omega'_q\hat{b}'^\dagger\hat{b}' + \Omega'_r\hat{a}'^\dagger\hat{a}'\,,$$

where the $'$ superscript describes the dressed frequencies and dressed operators. In the limit where $g \gg |\Omega_q - \Omega_r|$, $\hat{b} \approx \hat{b}' + \epsilon\hat{a}'$, where $\epsilon \propto g/(\Omega_q - \Omega_r)$. Injecting this expression in the Josephson cosine term, and expanding the cosine to it's Taylor series, yields many nonlinear coupling terms between these dressed modes.