Exam in Circuit QED, M2 ICFP
The asymmetric SQUID Transmon

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This exercise is about a widely employed qubit: the asymmetric SQUID Transmon. Its circuit is displayed in Fig. 1, and the goal here is to compute its transition frequency and understand its strengths and weaknesses. An extensive experimental work on this qubit can be found in Ref [Phys. Rev. App. 8, 044003 (2017)]. This type of qubit was used in a recent breakthrough experiment on quantum error correction [Nature volume 605, pages 669–674 (2022)].

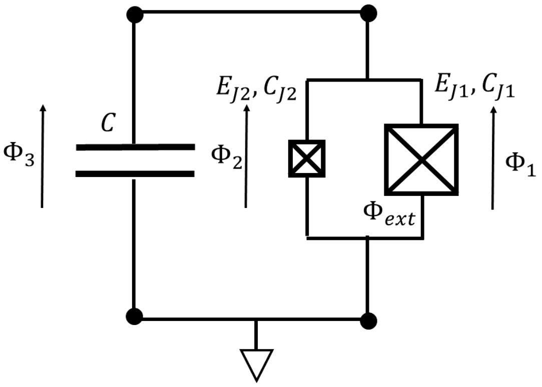

Figure 1: Circuit diagram of an asymmetric SQUID Transmon.

1. Consider the circuit displayed in Fig. 1. Write down the constraints that link the degrees of freedom Φ_1, Φ_2, Φ_3 and the external flux Φ_{ext}. Show that these constraints are fulfilled by the following parametrization

$$\Phi_1 = \Phi + \Phi_{\text{ext}}/2 \tag{1}$$
$$\Phi_2 = \Phi - \Phi_{\text{ext}}/2 \tag{2}$$
$$\Phi_3 = \Phi - \Phi_{\text{ext}}/2 \tag{3}$$

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The Kirchoff law for branch fluxes in both loops yields

$$\begin{aligned} \Phi_1 - \Phi_2 &= \Phi_{\text{ext}} \\ \Phi_2 - \Phi_3 &= 0\,. \end{aligned}$$

The proposed parametrization fulfills these two equations.

2. Compute the kinetic energy $T(\dot{\Phi})$ of this circuit. Use the notation $C_\Sigma = C + C_{J1} + C_{J2}$. Deduce the expression of the charge Q.

   Each capacitance contributes to the kinetic energy of the circuit.

   $$\begin{aligned} T(\dot{\Phi}) &= \frac{C}{2}\dot{\Phi}_3^2 + \frac{C_{J2}}{2}\dot{\Phi}_2^2 + \frac{C_{J1}}{2}\dot{\Phi}_1^2 \\ &= \frac{C + C_{J1} + C_{J2}}{2}\dot{\Phi}^2 \end{aligned}$$

   This yields

   $$\boxed{T(\dot{\Phi}) = \frac{C_\Sigma}{2}\dot{\Phi}^2}$$

   The charge degree of freedom is defined from the Lagrangian L as $Q = \frac{\partial L}{\partial \dot{\Phi}} = \frac{\partial T}{\partial \dot{\Phi}}$. This yields

   $$\boxed{Q = C_\Sigma \dot{\Phi}}$$

3. Compute the potential energy $U(\Phi)$ of this circuit. Use the notation $E_{J1} = E_J + \delta E_J$, and $E_{J2} = E_J - \delta E_J$. We will assume that $E_{J2} \ll E_{J1}$ so that δE_J and E_J are both of order $E_{J1}/2$.

   The potential energy of the circuit stems from the Josepshon energies of the two junctions.

   $$\begin{aligned} U(\Phi) &= -E_{J1}\cos(2\pi\Phi_1/\Phi_0) - E_{J2}\cos(2\pi\Phi_2/\Phi_0) \\ &= -E_{J1}\cos(2\pi(\Phi + \Phi_{\text{ext}}/2)/\Phi_0) \\ &- E_{J2}\cos(2\pi(\Phi - \Phi_{\text{ext}}/2/\Phi_0) \\ &= -(E_J + \delta E_J)\cos(2\pi(\Phi + \Phi_{\text{ext}}/2)/\Phi_0) \\ &- (E_J - \delta E_J)\cos(2\pi(\Phi - \Phi_{\text{ext}}/2/\Phi_0) \\ &= -2E_J\cos(\pi\Phi_{\text{ext}}/\Phi_0)\cos(2\pi\Phi/\Phi_0) \\ &+ 2\delta E_J\sin(\pi\Phi_{\text{ext}}/\Phi_0)\sin(2\pi\Phi/\Phi_0) \\ &= -2E_J\cos(\pi\Phi_{\text{ext}}/\Phi_0)\left(\cos(2\pi\Phi/\Phi_0) - \frac{\delta E_J}{E_J}\tan(\pi\Phi_{\text{ext}}/\Phi_0)\sin(2\pi\Phi/\Phi_0)\right)\,. \end{aligned}$$

4. Show that $U(\Phi)$ may be written as

   $$U(\Phi) = -2E_J\cos(\pi\Phi_{\text{ext}}/\Phi_0)f_1(\Phi_{\text{ext}})\cos(2\pi\left(\Phi - f_2(\Phi_{\text{ext}})\right)/\Phi_0)\,,$$

where $f_1(\Phi_{\text{ext}})$ and $f_2(\Phi_{\text{ext}})$ are functions of the external flux that must be computed. Give a physical interpretation of $f_2(\Phi_{\text{ext}})$. What is the effective Josephson energy $E_{J,eff}(\Phi_{\text{ext}})$ of this circuit ? Recall that $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$.

We expand the cosine in the proposed expression and identify the terms with respect to the expression derived in the previous question :

$$\begin{aligned} f_1(\Phi_{\text{ext}})\cos(2\pi f_2(\Phi_{\text{ext}})/\Phi_0) &= 1 \\ f_1(\Phi_{\text{ext}})\sin(2\pi f_2(\Phi_{\text{ext}})/\Phi_0) &= -\frac{\delta E_J}{E_J}\tan(\pi\Phi_{\text{ext}}/\Phi_0)\,. \end{aligned}$$

Dividing the second equation by the first one yields

$$\tan(2\pi f_2(\Phi_{\text{ext}})/\Phi_0) = -\frac{\delta E_J}{E_J}\tan(\pi\Phi_{\text{ext}}/\Phi_0)$$

Additionally, $f_1(\Phi_{\text{ext}}) = 1/\cos(2\pi f_2(\Phi_{\text{ext}})/\Phi_0)$. Since $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, this yields

$$f_1(\Phi_{\text{ext}}) = \sqrt{1+\left(\frac{\delta E_J}{E_J}\tan(\pi\Phi_{\text{ext}}/\Phi_0)\right)^2}\,.$$

The term $f_2(\Phi_{\text{ext}})$ may be understood as the classical value of the flux Φ at equilibrium. The quantum treatment of the problem will consider quantum fluctuations around this equilibrium. Th effective Josephson energy is then

$$\boxed{E_{J,eff}(\Phi_{\text{ext}}) = 2E_J\cos(\pi\Phi_{\text{ext}}/\Phi_0)\sqrt{1+\left(\frac{\delta E_J}{E_J}\tan(\pi\Phi_{\text{ext}}/\Phi_0)\right)^2}} \quad (4)$$

We may observe that $E_{J,eff}$ takes its maximal and minimal value at $\Phi_{\text{ext}} = 0$ and $\Phi_{\text{ext}} = \Phi_0/2$ respectively :

$$\begin{aligned} E_{J,eff}(0) &= 2E_J \\ E_{J,eff}(\Phi_0/2) &= 2|\delta E_J|\,. \end{aligned}$$

Finally,

$$\boxed{U(\Phi) = -E_{J,eff}(\Phi_{\text{ext}})\cos(2\pi\left(\Phi - f_2(\Phi_{\text{ext}})\right)/\Phi_0)}$$

5. Derive the quantum Hamiltonian of this circuit. Use the notation $\hat{\varphi} = 2\pi(\hat{\Phi} - f_2(\Phi_{\text{ext}}))/\Phi_0$ and $\hat{N} = \hat{Q}/2e$.

   First, the Lagrangian L reads

   $$L(\Phi,\dot{\Phi}) = T(\dot{\Phi}) - U(\Phi)\,.$$

The Hamiltonian then reads

$$\begin{aligned} H(\Phi, Q) &= Q\dot{\Phi}(Q) - T(\dot{\Phi}(Q)) + U(\Phi) \\ &= \frac{Q^2}{C_\Sigma} - \frac{C_\Sigma}{2}\left(\frac{Q}{C_\Sigma}\right)^2 + U(\Phi) \\ &= \frac{Q^2}{2C_\Sigma} + U(\Phi) \ . \end{aligned}$$

The quantum Hamiltonian is obtained by promoting Φ and Q to operators

$$\hat{H}(\hat{\Phi}, \hat{Q}) = \frac{\hat{Q}^2}{2C_\Sigma} + U(\hat{\Phi}) \ .$$

In terms of the reduced variable $\hat{N}$ and $\hat{\varphi}$, this yields

$$\boxed{\hat{H} = 4E_C\hat{N}^2 - E_{J,eff}(\Phi_{\text{ext}})\cos(\hat{\varphi})} \tag{5}$$

where $E_C = e^2/2C_\Sigma$ and $E_{J,eff}$ is given in Eq. (4).

6. We place ourselves in the regime where the Josephson energy is much greater than the charging energy. By analogy to a circuit derived in class, compute the lowest two transition frequencies of this circuit as a function of external flux, and provide a qualitative plot.

   The Hamiltonian of Eq. (5) is simply that of a Cooper-pair box with a flux dependent Josephson energy. Since we have assumed $\delta E_J \approx E_J$ then $E_{J,eff} \approx 2E_J \approx 2\delta E_J$. Hence, in the regime $E_J \gg E_C$, we also have $E_{J,eff} \gg E_C$, and so our circuit is in the transmon regime. The first transition energy is then

$$\hbar\omega_{ge}(\Phi_{\text{ext}}) = \sqrt{8E_{J,eff}(\Phi_{\text{ext}})E_C} - E_C$$

   and the second transition energy is

$$\hbar\omega_{ef}(\Phi_{\text{ext}}) = \hbar\omega_{ge}(\Phi_{\text{ext}}) - E_C$$

   Some experimentally measured transition frequencies are displayed in Fig. 2.

7. Give an expression for the dephasing rate of this qubit versus external flux.

   When projected onto its two lowest eigenstates, the transmon qubit may be described by the following two-level system (TLS)

$$\hat{H}_{TLS} = \hbar\omega_{ge}(\Phi_{\text{ext}})\sigma_z/2 \ .$$

   Since we consider that we are in the Transmon regime, charge noise is exponentially suppressed in the ration of Josephson to charging energy

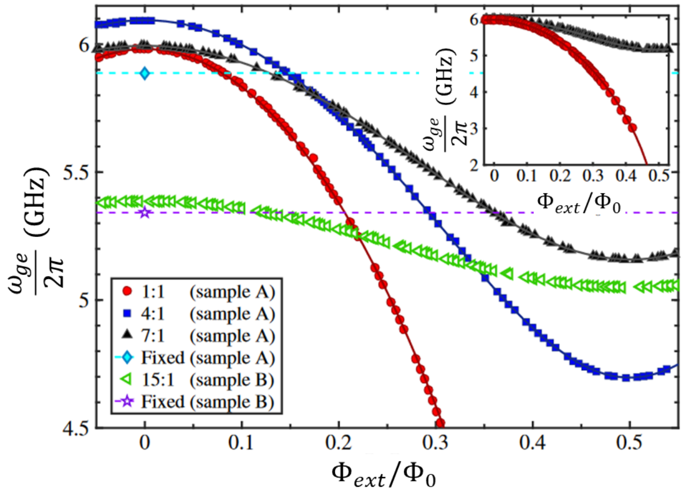

Figure 2: Figure extracted from Ref. [Phys. Rev. App. 8, 044003 (2017)]. Measured first transition frequency versus external flux. The different curves correspond to different rations E_{J1}/E_{J2}. For example, the red curve corresponds to $E_{J1} = E_{J2}$ (symmetric SQUID), the blue one to $E_{J1} = 4E_{J2}$, the black one to $E_{J1} = 7E_{J2}$ and the green one to $E_{J1} = 15E_{J2}$.

and can be disregarded. Dephasing will then occur dominantly through fluctuations $\delta\Phi_{\text{ext}}$ of the external flux. In the presence of fluctuations, the Hamiltonian reads

$$\hat{H}_{TLS}/\hbar = \omega_{ge}(\Phi_{\text{ext}})\sigma_z/2 + \frac{1}{2}\frac{\partial\omega_{ge}(\Phi_{\text{ext}})}{\partial\Phi_{\text{ext}}}\delta\Phi_{\text{ext}}\sigma_z \ .$$

We denote

$$\eta_z = \frac{1}{2}\frac{\partial\omega_{ge}(\Phi_{\text{ext}})}{\partial\Phi_{\text{ext}}}\delta\Phi_{\text{ext}} \ .$$

The dephasing rate is then given by

$$\Gamma_\varphi = 2S_{\eta_z\eta_z}(\omega = 0) \ ,$$

where $S_{\eta_z\eta_z}$ denotes the noise spectral density of η_z.

8. What are the benefits and drawbacks of this circuit versus the regular Transmon, or the symmetric SQUID Transmon ?

   With respect to a regular Transmon, this qubit has an adjustable frequency. This is a key asset to avoid frequency crowding on multi-qubit chips, perform two-qubit gates, and avoid cross talk. The drawback with respect to a regular Transmon is that the dephasing rate is expected to be larger due to fluctuations of the external flux. With respect to a symmetric SQUID Transmon, we have lost some tunability range (see Fig. 2), but on the other hand we now have two sweetspots instead of one: $\frac{\partial\omega_{ge}(\Phi_{\text{ext}})}{\partial\Phi_{\text{ext}}}$ vanishes at $\Phi_{\text{ext}} = 0$ and $\Phi_0/2$. Also the frequency flux dependence is less abrupt so that the dephasing rate is expected to be smaller.